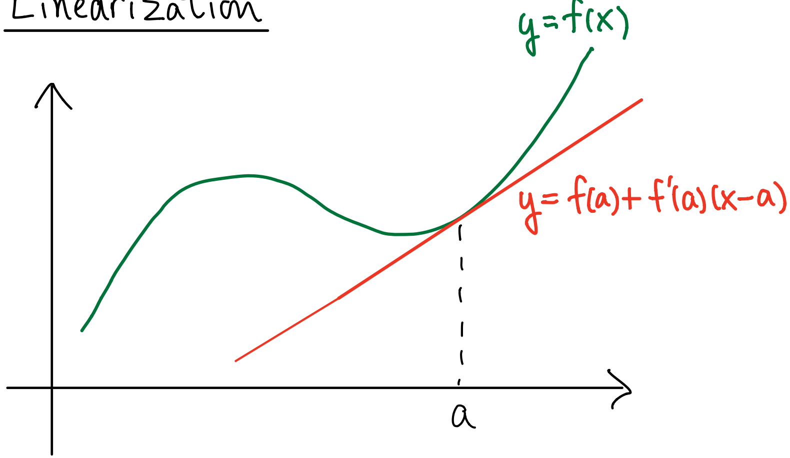
Math 1510 Week 6

## Linearization




Tangent at $a$ = "Best" straight line to approximate graph of $f(x)$ near $a$

**Linearization of $f(x)$ at $a$**

$$L(x) = f(a) + f'(a)(x-a)$$
$$\approx f(x)$$

eg Estimate $\sqrt{3.9}$ by linearization

Sol Let $f(x) = \sqrt{x}$, $\sqrt{3.9} = f(3.9)$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Pick $a = 4$. For $x$ near $4$,

$$\begin{aligned} f(x) &\approx L(x) \\ &= f(4) + f'(4)(x-4) \\ &= 2 + \tfrac{1}{4}(x-4) \end{aligned}$$

$$\begin{aligned} \sqrt{3.9} &= f(3.9) \\ &\approx 2 + \tfrac{1}{4}(3.9-4) \\ &= 2 - 0.025 \\ &= 1.975 \end{aligned}$$

Compare : $\sqrt{3.9} = 1.974841\ldots$

# Mean Value Theorems

## Rolle's Theorem

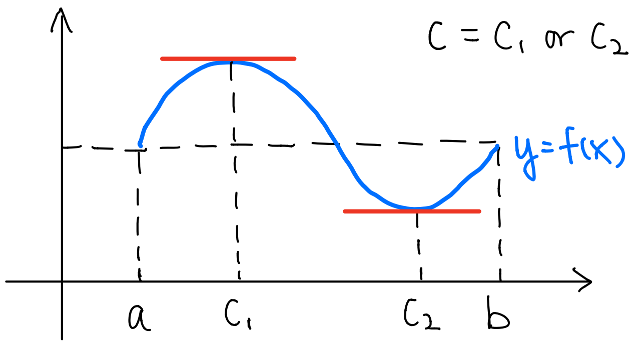
Let $f$ be continuous on $[a,b]$

differentiable on $(a,b)$.

Also $f(a)=f(b)$.

Then $\exists c \in (a,b)$ such that

$$f'(c)=0$$



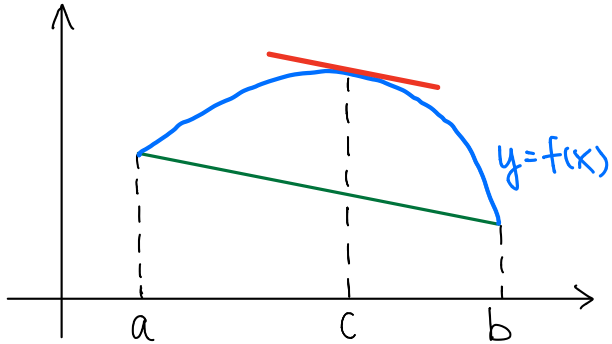

## Lagrange's Mean Value Theorem

Let $f$ be continuous on $[a,b]$

differentiable on $(a,b)$.

Then $\exists c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$




Rmk For the special case $f(a)=f(b)$,

MVT reduces to Rolle's Theorem

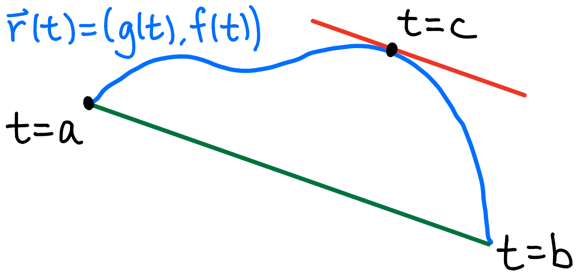
## Cauchy's Mean Value Theorem

Let $f, g$ be continuous on $[a,b]$
differentiable on $(a,b)$.

Also, $g'(x) \neq 0$ on $(a,b)$.

Then $\exists\, c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$




Rmk Cauchy's MVT can be used to prove L'Hopital Rule

eg Apply Lagrange's MVT to $f(x) = \arctan x$ on $[3,4]$ to show

$$\arctan 3 + \frac{1}{17} < \arctan 4 < \arctan 3 + \frac{1}{10} \quad (*)$$

Sol $f$ is continuous and differentiable on $[3,4]$

By Lagrange's MVT, $\exists\, c \in (3,4)$ such that

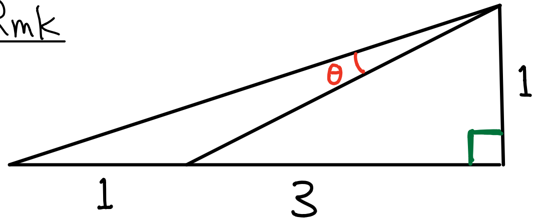
$$f'(c) = \frac{f(4)-f(3)}{4-3}$$

$$\frac{1}{1+c^2} = \arctan 4 - \arctan 3$$

$$3 < c < 4 \Rightarrow \frac{1}{1+4^2} < \frac{1}{1+c^2} < \frac{1}{1+3^2}$$

$$\therefore \frac{1}{17} < \arctan 4 - \arctan 3 < \frac{1}{10} \Rightarrow (*)$$

Rmk




We showed

$$\frac{1}{17} < \theta < \frac{1}{10}$$

Another application of MVT:

## L'Hopital's Rule

Let $a \in \mathbb{R}$ or $\pm\infty$.

Suppose $f, g$ are differentiable near $a$. Also,

i. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or

$\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$

ii. $g'(x) \neq 0$ for $x$ near $a$ (but $x \neq a$)

iii. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or $\pm\infty$

Then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

**Rmk**

① It can be proved by Cauchy's MVT

② Similar result for one-side limit.

**eg**

①
$$\lim_{x\to 1} \frac{x - e^{x-1}}{(x-1)^2} \quad \left(\frac{0}{0}\right)$$

L'Hopital ($\frac{d}{dx}$ both top and bottom)

$$= \lim_{x\to 1} \frac{1 - e^{x-1}}{2(x-1)} \quad \left(\frac{0}{0}\right)$$

L'Hopital again

$$= \lim_{x\to 1} \frac{-e^{x-1}}{2}$$

$$= \frac{-e^{1-1}}{2}$$

$$= -\frac{1}{2}$$

② $\lim\limits_{x\to\infty} \dfrac{e^{2x}}{x^2+4x+1}$ $\left(\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty} \dfrac{2e^{2x}}{2x+4}$ $\left(\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty} \dfrac{4e^{2x}}{2}$

$= \lim\limits_{x\to\infty} 2e^{2x}$

$= \infty$ (DNE)

③ $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2+4x+1}$ $\left(\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{2x+4}$

$= \lim\limits_{x\to\infty} \dfrac{1}{x(2x+4)}$

$= 0$

**Rmk** As seen ②, ③ above, as $x\to\infty$

$$e^{2x} >> x^2+4x+1 >> \ln x \quad (>> \text{ means much greater})$$

In general, as $x\to\infty$, growth rate of

Exponential Functions > Polynomials > Logarithmic Functions

④ $\lim\limits_{x\to\infty} \dfrac{\sin x + x}{x}$ $\left(\dfrac{\infty}{\infty}\right)$

~~$=$~~ $\underbrace{\lim\limits_{x\to\infty} \dfrac{\cos x + 1}{1}}$

DNE, not $\pm\infty$

$\Rightarrow$ L'Hopital Rule does not apply

Correct Answer For $x > 0$,

$$\frac{-1+x}{x} \le \frac{\sin x + x}{x} \le \frac{1+x}{x}$$

$$\lim_{x\to\infty} \frac{-1+x}{x} = \lim_{x\to\infty} -\frac{1}{x} + 1 = 1$$

$$\lim_{x\to\infty} \frac{1+x}{x} = \lim_{x\to\infty} \frac{1}{x} + 1 = 1$$

By Sandwich theorem,

$$\lim_{x\to\infty} \frac{\sin x + x}{x} = 1$$

Standard form in L'Hopital's Rule: $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Variations: $\underbrace{0\cdot(\pm\infty),\ \infty-\infty,\ 1^{\pm\infty},\ \infty^0,\ 0^0}$

Stategy: Convert them to $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

⑤

$$\lim_{x\to 0^+} x\ln x \quad (0\cdot -\infty)$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \quad \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} -x$$

$$= 0$$

<u>Rmk</u> If we tried another way

$$\lim_{x\to 0^+} x\ln x$$

$$= \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}}$$

$$= \lim_{x\to 0^+} \frac{1}{\frac{-1}{x(\ln x)^2}}$$

$$= \lim_{x\to 0^+} \underbrace{-x(\ln x)^2}$$

Even more complicated

∴ Not good!

⑥

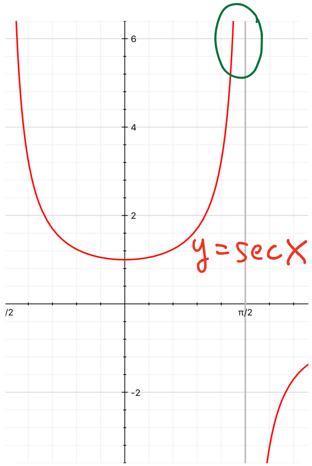
$$\lim_{x\to \frac{\pi}{2}^-} \sec x - \tan x \quad (\infty-\infty)$$

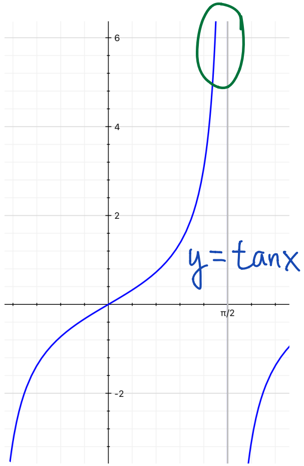
$$= \lim_{x\to \frac{\pi}{2}^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$

$$= \lim_{x\to \frac{\pi}{2}^-} \frac{1-\sin x}{\cos x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to \frac{\pi}{2}^-} \frac{-\cos x}{-\sin x}$$

$$= \frac{-\cos\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = 0$$



⑦ $\lim_{x\to 0} (\cos x)^{\csc x}$ $\left(1^{\pm\infty}\right)$

Sol Let $y = (\cos x)^{\csc x}$

$$\ln y = \csc x \ln \cos x$$

$$\therefore \lim_{x\to 0} \ln y = \lim_{x\to 0} \csc x \ln \cos x \quad (\pm\infty \cdot 0)$$

$$= \lim_{x\to 0} \frac{\ln \cos x}{\sin x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{\cos x}$$

$$= \lim_{x\to 0} -\frac{\sin x}{\cos^2 x}$$

$$= -\frac{\sin 0}{\cos^2 0} = 0$$

$$\therefore \lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\lim_{x\to 0} \ln y} = e^0 = 1$$

$e^z$ is continuous in $z$

⑧ $\lim_{x\to\infty} x^{\frac{1}{x}}$ $\left(\infty^0\right)$

Sol Let $y = x^{\frac{1}{x}}$

$$\ln y = \frac{1}{x} \ln x$$

$$\therefore \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= 0$$

$$\therefore \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^0 = 1$$

$e^z$ is continuous in $z$

> Recall: If $f(x)$ is continuous, then
> $$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

⑨ $\lim_{x\to 0^+} (1-\cos x)^{\frac{1}{\ln x}}$ $(0^0)$

Let $y=(1-\cos x)^{\frac{1}{\ln x}}$, then $\ln y = \frac{1}{\ln x}\ln(1-\cos x)$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1-\cos x)}{\ln x} \quad \left(\frac{-\infty}{-\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{\sin x}{1-\cos x}}{\frac{1}{x}}$$

$$= \lim_{x\to 0^+} \frac{x\sin x}{1-\cos x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0^+} \frac{\sin x + x\cos x}{\sin x}$$

$$= \lim_{x\to 0^+} \left(1 + \frac{x}{\sin x}\cos x\right)$$

$$= 1 + (1)(\cos 0) = 2$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+}\ln y} = e^2$$